

Differential Geometry and Its Applications.

## $\delta(3)$ -IDEAL NULL 2-TYPE HYPERSURFACES IN EUCLIDEAN SPACES

BANG-YEN CHEN<sup>1</sup> AND YU FU<sup>2</sup>

<sup>1</sup> *Department of Mathematics, Michigan State University,  
East Lansing, Michigan 48824-1027, USA*

<sup>2</sup> *School of Mathematics and Quantitative Economics,  
Dongbei University of Finance and Economics,  
Dalian 116025, P. R. China*

**ABSTRACT.** In the theory of finite type submanifolds, null 2-type submanifolds are the most simple ones, besides 1-type submanifolds (cf. e.g., [3, 12]). In particular, the classification problems of null 2-type hypersurfaces are quite interesting and of fundamentally important. In this paper, we prove that every  $\delta(3)$ -ideal null 2-type hypersurface in a Euclidean space has constant mean curvature and constant scalar curvature.

### 1. INTRODUCTION

In the late 1970s, the first author [12] introduced the theory of finite type submanifolds in order to derive the best possible estimates of the total mean curvature of a compact submanifold of Euclidean space in terms of spectral geometry. Since then the theory of finite type has been developed greatly (see [12] for more details).

Let  $x : M^n \rightarrow \mathbb{E}^m$  be an isometric immersion of an  $n$ -dimensional connected Riemannian manifold  $M^n$  into the Euclidean  $m$ -space  $\mathbb{E}^m$ . Denote by  $\Delta$  the Laplace operator with respect to the induced Riemannian metric. A submanifold  $M^n$  of  $\mathbb{E}^m$  is said to be of *finite type* [5, 6, 12, 14] if its position vector field  $x$  admits the following spectral decomposition:

$$x = c_0 + x_1 + \cdots + x_k,$$

where  $c_0$  is a constant vector and  $x_1, \dots, x_k$  are non-constant maps satisfying

$$\Delta x_i = \lambda_i x_i, \quad i = 1, \dots, k.$$

In particular, if all of the eigenvalues  $\lambda_1, \dots, \lambda_k$  are mutually different, then the submanifold  $M^n$  is said to be of  $k$ -type. In particular, if one of  $\lambda_1, \dots, \lambda_k$  is zero, then  $M^n$  is said to be of null  $k$ -type. Obviously, null  $k$ -type immersions occur only when  $M^n$  is non-compact.

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Email addresses: bychen@math.msu.edu, yufudufe@gmail.com.

It is well-known that a 1-type submanifold of a Euclidean space  $\mathbb{E}^m$  is either a minimal submanifold of  $\mathbb{E}^m$  or a minimal submanifold of a hypersphere in  $\mathbb{E}^m$ .

By the definition, null 2-type submanifolds are the most simple ones of finite type submanifolds besides 1-type submanifolds. After choosing a coordinate system on  $\mathbb{E}^m$  with  $c_0$  as its origin, we have the following simple spectral decomposition for a null 2-type submanifold  $M^n$ :

$$x = x_1 + x_2, \quad \Delta x_1 = 0, \quad \Delta x_2 = ax_2, \quad (1.1)$$

where  $a$  is non-zero real number. According to the well-known Beltrami formula  $\Delta x = -n\vec{H}$ , (1.1) implies the following equation

$$\Delta \vec{H} = a\vec{H}, \quad (1.2)$$

where  $\vec{H}$  is the mean curvature vector. Biharmonic submanifolds in  $\mathbb{E}^m$  are defined by the equation  $\Delta \vec{H} = 0$ . A result from [4] states that a Euclidean submanifold satisfying (1.2) is either biharmonic, or of 1-type, or of null 2-type.

Due to its simplicity, the first author proposed in 1991 the following problem [5, Problem 12]:

*“Determine all submanifolds of Euclidean spaces which are of null 2-type. In particular, classify null 2-type hypersurfaces in Euclidean spaces.”*

The first result on null 2-type submanifolds was obtained by the first author in 1988 by proving that every null 2-type surface in  $\mathbb{E}^3$  is an open portion of a circular cylinder  $S^1 \times \mathbb{R}$  [3]. Later on, Ferrández and Lucas [19] showed that a null 2-type hypersurface in  $\mathbb{E}^{n+1}$  with at most two distinct principal curvatures is a spherical cylinder  $S^p \times \mathbb{R}^{n-p}$ . In 1995, Hasanis and Vlachos [21] proved that null 2-type hypersurfaces in  $\mathbb{E}^4$  have constant mean curvature and constant scalar curvature (see also [16]). In 2012, the first author and Garraý [13] proved that  $\delta(2)$ -ideal null 2-type hypersurfaces in Euclidean space are spherical cylinders. In addition,  $\delta(2)$ -ideal  $H$ -hypersurfaces of a Euclidean space were classified by the first and Munteanu in [15]. In [23], Turgay determined  $H$ -hypersurfaces in a Euclidean space with three distinct principal curvatures. Very recently, the second author proved in [20] that null 2-type hypersurfaces with at most three distinct principal curvatures have constant mean curvature and constant scalar curvature. Null 2-type submanifolds with codimension  $\geq 2$  have been studied, among others, in [14, 17, 18]. For the most recent surveys in this field, we refer the readers to [11, 12].

In 1991, the first author posted in [5] the following challenging conjecture:

*The only biharmonic submanifolds of Euclidean spaces are the minimal ones.*

Since then biharmonic submanifolds become a very active research subject (cf. [10–12]). However, this biharmonic conjecture remains open.

For an  $n$ -dimensional Riemannian manifold  $M^n$  with  $n \geq 3$  and an integer  $r \in [2, n-1]$ , the first author introduced the  $\delta$ -invariant  $\delta(r)$  by

$$\delta(r)(p) = \tau(p) - \inf \tau(L_p^r), \quad (1.3)$$

where  $\tau(p)$  is the scalar curvature of  $M^n$  and  $\inf \tau(L_p^r)$  is the function assigning to the point  $p$  the infimum of the scalar curvature for  $L_p^r$  running over all  $r$ -dimensional linear subspaces in  $T_p M^n$  (cf. [9] for details).

For any isometric immersion of a Riemannian  $n$ -manifold  $M^n$  ( $n \geq 3$ ) into a Euclidean  $m$ -space  $\mathbb{E}^m$ , the first author proved the following universal inequality [8]:

$$\delta(r) \leq \frac{n^2(n-r)}{2(n-r+1)} H^2, \quad (1.4)$$

where  $H^2$  is the squared mean curvature.

Since the inequality (1.4) is a very general and sharp inequality, it is a very natural and interesting problem to classify submanifolds satisfying the equality case of (1.4) identically. Following [9], such a submanifold in  $\mathbb{E}^m$  is called  $\delta(r)$ -ideal.  $\delta(2)$  and  $\delta(3)$ -ideal submanifolds are the simplest ideal submanifolds. Investigating the classification problems of  $\delta(2)$ -ideal and  $\delta(3)$ -ideal submanifolds is quite interesting. In particular, many interesting results on  $\delta(2)$ -ideal submanifolds has been done by many geometers since the invention of  $\delta$ -invariants (see [9] for details, and recent work [1, 22]). In contrast, few results on  $\delta(3)$ -ideal submanifolds are known.

In this paper, we investigate  $\delta(3)$ -ideal null 2-type hypersurfaces in Euclidean space. Our main result states that every  $\delta(3)$ -ideal null 2-type hypersurface in a Euclidean space has constant mean curvature and constant scalar curvature.

## 2. PRELIMINARIES

Let  $x : M^n \rightarrow \mathbb{E}^{n+1}$  be an isometric immersion of a hypersurface  $M^n$  into  $\mathbb{E}^{n+1}$ . Denote the Levi-Civita connections of  $M^n$  and  $\mathbb{E}^{n+1}$  by  $\nabla$  and  $\bar{\nabla}$ , respectively. Let  $X$  and  $Y$  be vector fields tangent to  $M^n$  and let  $\xi$  be a unit normal vector field. Then the Gauss and Weingarten formulas are given respectively by (cf. [2, 9])

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.1)$$

$$\bar{\nabla}_X \xi = -AX, \quad (2.2)$$

where  $h$  is the second fundamental form, and  $A$  is the shape operator (or the Weingarten operator). It is well known that the second fundamental form  $h$  and the shape operator  $A$  are related by

$$\langle h(X, Y), \xi \rangle = \langle AX, Y \rangle. \quad (2.3)$$

The mean curvature vector field  $\vec{H}$  is given by

$$\vec{H} = \left( \frac{1}{n} \right) \text{Tr } h, \quad (2.4)$$

where  $\text{Tr } h$  is the trace of  $h$ . The Gauss and Codazzi equations are given respectively by

$$R(X, Y)Z = \langle AY, Z \rangle AX - \langle AX, Z \rangle AY,$$

$$(\nabla_X A)Y = (\nabla_Y A)X,$$

where  $R$  is the curvature tensor,  $\langle \cdot, \cdot \rangle$  the inner product, and  $\nabla A$  is defined by

$$(\nabla_X A)Y = \nabla_X (AY) - A(\nabla_X Y) \quad (2.5)$$

for all  $X, Y, Z$  tangent to  $M$ . Let us put  $\vec{H} = H\xi$ , where  $H$  denotes the mean curvature. The scalar curvature  $\tau$  is then given by

$$\tau = \frac{1}{2}(n^2 H^2 - \text{Tr } A^2). \quad (2.6)$$

By identifying the tangent and the normal parts of the condition (1.2), we have the following necessary and sufficient conditions for  $M^n$  to be of null 2-type in  $\mathbb{E}^{n+1}$  (cf. e.g. [7, 12, 13, 15]).

**Proposition 1.** *Assume that  $M^n$  is not of 1-type. A hypersurface  $M^n$  in a Euclidean  $(n+1)$ -space  $\mathbb{E}^{n+1}$  is null 2-type if and only if*

$$\begin{cases} \Delta H + H \text{Tr } A^2 = aH, \\ 2A \text{grad} H + n H \text{grad} H = 0, \end{cases} \quad (2.7)$$

where the Laplace operator  $\Delta$  acting on scalar-valued function  $f$  is given by

$$\Delta f = - \sum_{i=1}^n (e_i e_i f - \nabla_{e_i} e_i f) \quad (2.8)$$

for an orthonormal local tangent frame  $\{e_1, \dots, e_n\}$  on  $M^n$ .

We need the following result from [9, Theorem 13.7].

**Proposition 2.** *Let  $M^n$  be a hypersurface in the Euclidean space  $\mathbb{E}^{n+1}$ . Then*

$$\delta(3) \leq \frac{n^2(n-3)}{2(n-2)} H^2, \quad (2.9)$$

where the equality case holds at a point  $p$  if and only if there is an orthonormal basis  $\{e_1, \dots, e_n\}$  at  $p$  such that the shape operator at  $p$  satisfies

$$A = \begin{pmatrix} \alpha & 0 & 0 & 0 & \dots & 0 \\ 0 & \beta & 0 & 0 & \dots & 0 \\ 0 & 0 & \gamma & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha + \beta + \gamma & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha + \beta + \gamma \end{pmatrix} \quad (2.10)$$

for some functions  $\alpha, \beta, \gamma$  defined on  $M^n$ . If this happens at every point, we call  $M^n$  a  $\delta(3)$ -ideal hypersurface in  $\mathbb{E}^{n+1}$ .

### 3. $\delta(3)$ -IDEAL NULL 2-TYPE HYPERSURFACES

In this section, we determine  $\delta(3)$ -ideal null 2-type hypersurfaces  $M^n$  in Euclidean space  $\mathbb{E}^{n+1}$  with  $n \geq 4$ . We assume that  $M^n$  is not of 1-type, hence  $M^n$  is not minimal.

If the mean curvature  $H$  is constant, the first equation of (2.7) implies that the length of the second fundamental form is also constant. Combining these with (2.6) shows that the scalar curvature  $\tau$  is constant as well. Hence, in the following text we suppose that the mean curvature  $H$  is non-constant.

**Lemma 3.1.** *Let  $M^n$  be a  $\delta(3)$ -ideal hypersurface satisfying the second equation of (2.7) in  $\mathbb{E}^{n+1}$  with non-constant mean curvature  $H$ . If the shape operator of  $M^n$  satisfies (2.10), then, up to reordering of  $\alpha$ ,  $\beta$  and  $\gamma$ , with respect to a suitable orthonormal frame  $\{e_1, \dots, e_n\}$  we have*

$$\alpha = -\frac{n}{2}H \text{ and } \gamma = \frac{n^2}{2(n-2)}H - \beta.$$

*Proof.* Let  $M^n$  be a hypersurface satisfying the second equation in (2.7) with the shape operator given by (2.10). Also assume that  $\text{grad } H$  is non-vanishing. Then one of its principal curvatures  $\lambda_1$  must be  $-nH/2$  with multiplicity 1 and the corresponding principal direction is  $e_1 = \text{grad } \lambda_1 / |\text{grad } \lambda_1|$ . By taking into account (2.10), up to reordering  $\alpha, \beta, \gamma$  we can assume either  $\lambda_1 = \alpha$  or  $\lambda_1 = \alpha + \beta + \gamma$ . The former case gives the case of Lemma 3.1. In the latter case, because the multiplicity of  $\lambda_1$  is 1, we have  $n = 4$  which implies  $\alpha + \beta + \gamma = -2H$ . However, from these equations and (2.10) one can obtain

$$4H = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2(\alpha + \beta + \gamma) = -4H,$$

which implies  $H = 0$ . This is a contradiction.  $\square$

According to Lemma 3.1,  $e_1$  is parallel to  $\text{grad } H$  and so (2.10) becomes

$$A = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots, \lambda_n) \quad (3.1)$$

with  $\lambda_1 = -\frac{n}{2}H$ ,  $\lambda_2 = \beta$ ,  $\lambda_3 = \frac{n^2}{2(n-2)}H - \beta$  and  $\lambda_4 = \dots = \lambda_n = \frac{n}{n-2}H$ .

Denote by  $c_1 = -\frac{n}{2}$  and  $c_2 = \frac{n^2}{2(n-2)}$ , and hence  $\frac{n}{n-2} = c_1 + c_2$ . Note that in our case  $M^n$  has four distinct principal curvatures. Thus we have

$$\beta \neq c_1 H, \quad \frac{1}{2}c_2 H, \quad (c_1 + c_2)H, \quad (c_2 - c_1)H, \quad -c_1 H. \quad (3.2)$$

Since the vector field  $e_1$  is parallel to  $\text{grad } H$ , computing  $\text{grad } H = \sum_{i=1}^n e_i(H)e_i$  gives

$$e_1(H) \neq 0, \quad e_i(H) = 0, \quad 2 \leq i \leq n. \quad (3.3)$$

Let us put

$$\nabla_{e_i} e_j = \sum_{k=1}^n \omega_{ij}^k e_k, \quad 1 \leq i, j \leq n.$$

By computing  $\nabla_{e_k} \langle e_i, e_i \rangle = 0$  and  $\nabla_{e_k} \langle e_i, e_j \rangle = 0$ , we find

$$\omega_{ki}^i = 0, \quad (3.4)$$

$$\omega_{ki}^j + \omega_{kj}^i = 0, \quad (3.5)$$

for  $i \neq j$  and  $1 \leq i, j, k \leq n$ . From (3.1), (3.3) and (3.4), the Codazzi equation reduces to

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \quad (3.6)$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j, \quad (3.7)$$

where  $i, j, k$  are mutually different and  $1 \leq i, j, k \leq n$ . From (3.3) we have

$$[e_i, e_j](H) = 0, \quad 2 \leq i, j \leq n, \quad i \neq j,$$

which implies that

$$\omega_{ij}^1 = \omega_{ji}^1, \quad 2 \leq i, j \leq n, \quad i \neq j. \quad (3.8)$$

Compute  $[e_1, e_i](H) = (\nabla_{e_1} e_i - \nabla_{e_i} e_1)(H)$  for  $i = 2, \dots, n$ . From (3.3) we find  $\omega_{i1}^1 = 0$ . By choosing  $j = 1, i = 2, \dots, n$  in (3.5), and by (3.3) we obtain  $\omega_{1i}^1 = 0$  for  $2 \leq i \leq n$ . Hence we have

$$e_i e_1(H) = 0, \quad 2 \leq i \leq n. \quad (3.9)$$

Sequentially, by (3.9) the formula  $[e_1, e_i](e_1(H)) = (\nabla_{e_1} e_i - \nabla_{e_i} e_1)(e_1(H))$  gives

$$e_i e_1 e_1(H) = 0, \quad 2 \leq i \leq n. \quad (3.10)$$

By choosing  $j = 1$  in (3.7), we have

$$(\lambda_i - \lambda_1)\omega_{ki}^1 = (\lambda_k - \lambda_1)\omega_{ik}^1,$$

which together with (3.8) yields

$$\omega_{ij}^1 = 0, \quad 2 \leq i, j \leq n, \quad i \neq j. \quad (3.11)$$

Combining (3.11) with (3.5) gives

$$\omega_{i1}^j = 0, \quad 2 \leq i, j \leq n, \quad i \neq j. \quad (3.12)$$

By choosing  $k = 2$  in (3.7), we deduce that

$$(\lambda_i - \lambda_j)\omega_{2i}^j = (\lambda_2 - \lambda_j)\omega_{i2}^j,$$

which yields

$$\omega_{i2}^j = 0, \quad 4 \leq i, j \leq n, \quad i \neq j. \quad (3.13)$$

Similarly, we also have

$$\omega_{i3}^j = 0, \quad 4 \leq i, j \leq n, \quad i \neq j. \quad (3.14)$$

Now we state an important lemma for later use.

**Lemma 3.2.** *Under the assumptions above, we have*

$$e_i(\beta) = 0, \quad 2 \leq i \leq n.$$

*Proof.* By applying (3.1), we deduce from the Gauss equation that  $R(e_2, e_k)e_1 = 0$  for  $3 \leq k \leq n$ . It follows from (3.12) that

$$\begin{aligned} \nabla_{e_2} \nabla_{e_k} e_1 &= e_2(\omega_{k1}^k) e_k + \omega_{k1}^k \left( \sum_{i=2}^n \omega_{2k}^i e_i \right), \\ \nabla_{e_k} \nabla_{e_2} e_1 &= e_k(\omega_{21}^2) e_2 + \omega_{21}^2 \left( \sum_{i=3}^n \omega_{k2}^i e_i \right), \\ \nabla_{[e_2, e_k]} e_1 &= \sum_{i=2}^n (\omega_{2k}^i - \omega_{k2}^i) \omega_{i1}^i e_i. \end{aligned}$$

Hence, by the definition of curvature tensor,  $\langle R(e_2, e_k)e_1, e_2 \rangle = 0$  gives

$$e_k(\omega_{21}^2) = \omega_{2k}^2(\omega_{k1}^k - \omega_{21}^2), \quad 3 \leq k \leq n. \quad (3.15)$$

In a similar way, by taking into account  $\langle R(e_3, e_k)e_1, e_3 \rangle$ , we find

$$e_k(\omega_{31}^3) = \omega_{3k}^3(\omega_{k1}^k - \omega_{31}^3), \quad k = 2 \text{ or } 4 \leq k \leq n. \quad (3.16)$$

At this moment, by using (2.8), (3.1), (3.3), (3.5) and (3.6), the first equation of (2.7) becomes

$$\begin{aligned} & -e_1e_1(H) - \left\{ \frac{e_1(\beta)}{c_1H - \beta} + \frac{e_1(c_1H - \beta)}{(c_1 - c_2)H + \beta} - (n-3)\frac{(c_1 + c_2)e_1(H)}{c_2H} \right\} e_1(H) \\ & + H\{c_1^2H^2 + \beta^2 + (c_2H - \beta)^2 + (n-3)(c_1 + c_2)^2H^2 - a\} = 0. \end{aligned} \quad (3.17)$$

We first show that  $e_3(\beta) = 0$  and  $e_2(\beta) = 0$ .

By using (3.5), (3.6) and choosing  $k = 3$  in (3.15), (3.15) becomes

$$e_3\left(\frac{e_1(\beta)}{c_1H - \beta}\right) = \frac{e_3(\beta)}{c_2H - 2\beta} \left( \frac{e_1(c_2H - \beta)}{(c_1 - c_2)H + \beta} - \frac{e_1(\beta)}{c_1H - \beta} \right), \quad (3.18)$$

which can be rewritten as

$$e_3e_1(\beta) = \left\{ -\frac{e_1(\beta)}{c_1H - \beta} + \frac{c_1H - \beta}{c_2H - 2\beta} \left( \frac{e_1(c_2H - \beta)}{(c_1 - c_2)H + \beta} - \frac{e_1(\beta)}{c_1H - \beta} \right) \right\} e_3(\beta).$$

Hence

$$\begin{aligned} & e_3\left(\frac{e_1(c_1H - \beta)}{(c_1 - c_2)H + \beta}\right) \\ & = \frac{e_3(\beta)}{c_2H - 2\beta} \left( \frac{e_1(c_2H - \beta)}{(c_1 - c_2)H + \beta} - \frac{e_1(\beta)}{c_1H - \beta} \right) \frac{-(c_1 + c_2)H + 3\beta}{(c_1 - c_2)H + \beta}. \end{aligned} \quad (3.19)$$

Since  $e_3(\beta) \neq 0$ , by differentiating (3.17) along  $e_3$  and by applying (3.3), (3.9), (3.11), (3.18) and (3.19) we get

$$\frac{e_1(c_2H - \beta)}{(c_1 - c_2)H + \beta} - \frac{e_1(\beta)}{c_1H - \beta} + H(2\beta - c_2H)[(c_1 - c_2)H + \beta] = 0. \quad (3.20)$$

Also, by differentiating (3.20) along  $e_3$  and by using (3.18) and (3.19) we obtain

$$\left( \frac{e_1(c_2H - \beta)}{(c_1 - c_2)H + \beta} - \frac{e_1(\beta)}{c_1H - \beta} \right) \frac{-2c_1H + 2\beta}{(c_1 - c_2)H + \beta} + H(c_2H - 2\beta)[(2c_1 - 3c_2)H + 4\beta] = 0. \quad (3.21)$$

Thus (3.21) together with (3.20) implies

$$3H(2\beta - c_2H)^2 = 0,$$

which is equivalent to  $2\beta - c_2H = 0$ ; namely,

$$\lambda_2 = \lambda_3.$$

This contradicts to our assumption. Hence, we must have  $e_3(\beta) = 0$ . By applying (3.16) for  $k = 2$ , a quite similar argument as the above case yields  $e_2(\beta) = 0$ .

In the remaining case, we will prove  $e_k(\beta) = 0$  for  $k \geq 4$ .

After applying (3.6), equations (3.15) and (3.16) could be rewritten as

$$e_k\left(\frac{e_1(\beta)}{c_1H - \beta}\right) = -\frac{e_k(\beta)}{(c_1 + c_2)H - \beta} \left( \frac{(c_1 + c_2)e_1(H)}{c_2H} + \frac{e_1(\beta)}{c_1H - \beta} \right), \quad (3.22)$$

$$e_k\left(\frac{e_1(c_2H - \beta)}{(c_1 - c_2)H + \beta}\right) = \frac{e_k(\beta)}{c_1H + \beta} \left( \frac{(c_1 + c_2)e_1(H)}{c_2H} + \frac{e_1(c_2H - \beta)}{(c_1 - c_2)H + \beta} \right). \quad (3.23)$$

Assume  $e_k(\beta) \neq 0$ . Then after differentiating (3.17) along  $e_k$  and by applying (3.3), (3.9), (3.11), (3.22) and (3.23), we have

$$\begin{aligned} & \left\{ \frac{1}{(c_1 + c_2)H - \beta} \left( \frac{(c_1 + c_2)e_1(H)}{c_2H} + \frac{e_1(\beta)}{c_1H - \beta} \right) \right. \\ & \left. - \frac{1}{c_1H + \beta} \left( \frac{(c_1 + c_2)e_1(H)}{c_2H} + \frac{e_1(c_2H - \beta)}{(c_1 - c_2)H + \beta} \right) \right\} e_1(H) \\ & + 2H(2\beta - c_2H) = 0. \end{aligned} \quad (3.24)$$

Now, by differentiating (3.24) along  $e_k$ , we derive from (3.3), (3.9), (3.22) and (3.23) that  $4He_k(\beta) = 0$ , which is a contradiction. Consequently, we complete the proof of Lemma 3.2.  $\square$

**Lemma 3.3.** *The connection coefficients of the hypersurfaces  $M^n$  are expressed as follows:*

$$\begin{aligned} \nabla_{e_1}e_1 &= \nabla_{e_1}e_2 = \nabla_{e_1}e_3 = 0; \quad \nabla_{e_1}e_i = \sum_{k=4}^n \omega_{1i}^k e_k, \quad 4 \leq i \leq n; \\ \nabla_{e_i}e_1 &= \omega_{i1}^1 e_i, \quad 2 \leq i \leq n; \quad \nabla_{e_2}e_2 = \omega_{22}^1 e_1; \quad \nabla_{e_3}e_3 = \omega_{33}^1 e_1; \\ \nabla_{e_2}e_i &= \sum_{k=3}^n \omega_{2i}^k e_k, \quad 3 \leq i \leq n; \\ \nabla_{e_3}e_i &= \omega_{3i}^2 e_2 + \sum_{k=4}^n \omega_{3i}^k e_k, \quad i = 2, \text{ or } 3 \leq i \leq n; \\ \nabla_{e_j}e_2 &= \omega_{j2}^3 e_3, \quad 4 \leq j \leq n; \quad \nabla_{e_j}e_3 = \omega_{j3}^2 e_2, \quad 4 \leq j \leq n; \\ \nabla_{e_i}e_j &= \omega_{ij}^1 \delta_{ij} e_1 + \sum_{k=4}^n \omega_{ij}^k e_k, \quad 4 \leq i, j \leq n. \end{aligned}$$

*Proof.* By using (3.3)-(3.8), (3.11)-(3.14) and Lemma 3.2, a straightforward computation gives the connection coefficients of  $M^n$  in Lemma 3.3.  $\square$

By applying Gauss' equation and Lemma 3.3, we compute  $\langle R(X, Y)Z, W \rangle$ . Then we obtain successively:

- $X = Z = e_1, Y = W = e_2$ ,

$$e_1(\omega_{21}^2) + (\omega_{21}^2)^2 = -c_1 H \beta; \quad (3.25)$$

- $X = Z = e_1, Y = W = e_3$ ,

$$e_1(\omega_{31}^3) + (\omega_{31}^3)^2 = -c_1 H(c_2 H - \beta); \quad (3.26)$$

- $X = Z = e_1, Y = W = e_4$ ,

$$e_1(\omega_{41}^4) + (\omega_{41}^4)^2 = -c_1(c_1 + c_2)H^2; \quad (3.27)$$

- $X = Z = e_j, Y = W = e_2$  for  $4 \leq j \leq n$ ,

$$\omega_{jj}^1 \omega_{21}^2 - \omega_{2j}^3 \omega_{j3}^2 + (\omega_{j2}^3 - \omega_{2j}^3) \omega_{3j}^2 = (c_1 + c_2) \beta H; \quad (3.28)$$

- $X = Z = e_j, Y = W = e_3$  for  $4 \leq j \leq n$ ,

$$\omega_{jj}^1 \omega_{31}^3 - \omega_{3j}^2 \omega_{j2}^3 + (\omega_{j3}^2 - \omega_{3j}^2) \omega_{2j}^3 = (c_1 + c_2) H(c_2 H - \beta); \quad (3.29)$$



- $X = Z = e_3, Y = W = e_2,$

$$\omega_{33}^1 \omega_{21}^2 - \sum_{j=4}^n \omega_{23}^j \omega_{3j}^2 + \sum_{j=4}^n (\omega_{32}^j - \omega_{23}^j) \omega_{j3}^2 = \beta(c_2 H - \beta); \quad (3.30)$$

- $X = e_2, Y = e_j, Z = e_3, W = e_1$  for  $4 \leq j \leq n,$

$$\omega_{j3}^2 \omega_{22}^1 - \omega_{23}^j \omega_{jj}^1 - (\omega_{2j}^3 - \omega_{j2}^3) \omega_{33}^1 = 0;$$

- $X = e_3, Y = e_j, Z = e_2, W = e_1$  for  $4 \leq j \leq n,$

$$\omega_{j2}^3 \omega_{33}^1 - \omega_{32}^j \omega_{jj}^1 - (\omega_{3j}^2 - \omega_{j3}^2) \omega_{22}^1 = 0.$$

Moreover, it follows from (3.5) and (3.7) that

$$\omega_{23}^j = -\omega_{2j}^3 = -\frac{h_j}{\beta + c_1 H}, \quad (3.31)$$

$$\omega_{32}^j = -\omega_{3j}^2 = \frac{h_j}{\beta - (c_1 + c_2)H}, \quad (3.32)$$

$$\omega_{j2}^3 = -\omega_{j3}^2 = \frac{h_j}{2\beta - c_2 H}, \quad (3.33)$$

where  $h_j$  ( $4 \leq j \leq n$ ) are smooth functions defined on  $M^n$ . Therefore (3.31)-(3.33) imply that

$$\begin{aligned} -\omega_{2j}^3 \omega_{j3}^2 &= (\omega_{j2}^3 - \omega_{2j}^3) \omega_{3j}^2, \\ -\omega_{3j}^2 \omega_{j2}^3 &= (\omega_{j3}^2 - \omega_{3j}^2) \omega_{2j}^3, \\ -\omega_{23}^j \omega_{3j}^2 &= (\omega_{32}^j - \omega_{23}^j) \omega_{j3}^2. \end{aligned}$$

By combining (3.28), (3.29) and (3.30) with the above three equations, we get

$$\omega_{jj}^1 \omega_{21}^2 - 2\omega_{2j}^3 \omega_{j3}^2 = (c_1 + c_2)\beta H, \quad 4 \leq j \leq n, \quad (3.34)$$

$$\omega_{jj}^1 \omega_{31}^3 - 2\omega_{3j}^2 \omega_{j2}^3 = (c_1 + c_2)H(c_2 H - \beta), \quad 4 \leq j \leq n, \quad (3.35)$$

$$\omega_{33}^1 \omega_{21}^2 - 2 \sum_{j=4}^n \omega_{23}^j \omega_{3j}^2 = \beta(c_2 H - \beta). \quad (3.36)$$

Since  $\omega_{44}^1 = \omega_{55}^1 = \dots = \omega_{nn}^1$  and  $\omega_{2j}^3 \omega_{j3}^2 + \omega_{3j}^2 \omega_{j2}^3 + \omega_{23}^j \omega_{3j}^2 = 0$ , we derive from (3.34)-(3.36) that

$$\begin{aligned} (n-3)\omega_{jj}^1 \omega_{21}^2 + (n-3)\omega_{jj}^1 \omega_{31}^3 + \omega_{33}^1 \omega_{21}^2 \\ = (n-3)(c_1 + c_2)c_2 H^2 + \beta(c_2 H - \beta), \end{aligned} \quad (3.37)$$

By using (3.6), equations (3.25)-(3.27) can be rewritten, respectively, as

$$e_1 e_1(\beta) - c_1 \omega_{21}^2 e_1(H) + 2(c_1 H - \beta)(\omega_{21}^2)^2 = -c_1 H \beta(c_1 H - \beta), \quad (3.38)$$

$$e_1 e_1(c_2 H - \beta) - c_1 \omega_{31}^3 e_1(H) + 2\{(c_1 - c_2)H + \beta\}(\omega_{31}^3)^2 \quad (3.39)$$

$$= -c_1 H(c_2 H - \beta)\{(c_1 - c_2)H + \beta\},$$

$$(c_1 + c_2)e_1 e_1(H) - c_1 \omega_{41}^4 e_1(H) - 2c_2 H(\omega_{41}^4)^2 = c_1 c_2 (c_1 + c_2)H^3. \quad (3.40)$$

Also, it follows from (3.6) that

$$(c_1 H - \beta)\omega_{21}^2 + \{(c_1 - c_2)H + \beta\}\omega_{31}^3 = c_2 e_1(H), \quad (3.41)$$

$$-c_2 H \omega_{41}^4 = (c_1 + c_2)e_1(H). \quad (3.42)$$

Eliminating  $e_1 e_1(\beta)$  between (3.38) and (3.39) and applying (3.37), (3.41) and (3.42) give

$$\begin{aligned} & c_2^2 e_1 e_1(H) + \{2(n-3)(c_1 + c_2)(2c_1 - c_2) + c_2(2c_2 - c_1)\}(\omega_{21}^2 + \omega_{31}^3)e_1(H) \\ &= \{2(n-3)(2c_1 - c_2)(c_1 + c_2)c_2^2 - c_1 c_2^2(c_1 - c_2)\}H^3 \\ &+ 2c_2^2(c_1 - c_2)H^2\beta - 2c_2(c_1 - c_2)H\beta^2. \end{aligned} \quad (3.43)$$

Moreover, eliminating  $(\omega_{41}^4)^2$  in (3.40) by (3.42) we obtain

$$(c_1 + c_2)e_1 e_1(H) + (c_1 + 2c_2)\omega_{41}^4 e_1(H) = c_1 c_2(c_1 + c_2)H^3. \quad (3.44)$$

Note that (3.17) takes the form

$$\begin{aligned} & -e_1 e_1(H) - \{\omega_{21}^2 + \omega_{31}^3 + (n-3)\omega_{41}^4\}e_1(H) \\ &+ \{c_1^2 + c_2^2 + (n-3)(c_1 + c_2)^2\}H^3 - 2c_2 H^2\beta + 2H\beta^2 - aH = 0. \end{aligned} \quad (3.45)$$

By substituting  $c_1 = -\frac{n}{2}$  and  $c_2 = \frac{n^2}{2(n-2)}$  into (3.43)-(3.45), we get

$$e_1 e_1(H) - \frac{9n^2 - 50n + 48}{n^2}(\omega_{21}^2 + \omega_{31}^3)e_1(H) \quad (3.46)$$

$$+ \frac{n^2(7n^2 - 29n + 26)}{2(n-2)^2}H^3 + \frac{2n(n-1)}{n-2}H^2\beta - \frac{4(n-1)}{n}H\beta^2 = 0,$$

$$e_1 e_1(H) + \frac{n+2}{2}\omega_{41}^4 e_1(H) + \frac{n^3}{4(n-2)}H^3 = 0, \quad (3.47)$$

$$-e_1 e_1(H) - \{\omega_{21}^2 + \omega_{31}^3 + (n-3)\omega_{41}^4\}e_1(H) \quad (3.48)$$

$$+ \frac{n^2(n+2)}{2(n-2)}H^3 - \frac{n^2}{n-2}H^2\beta + 2H\beta^2 - aH = 0.$$

Eliminating  $e_1 e_1(H)$ , equations (3.46)-(3.48) reduce to

$$\begin{aligned} & \left\{ \frac{9n^2 - 50n + 48}{n^2}(\omega_{21}^2 + \omega_{31}^3) + \frac{n+2}{2}\omega_{41}^4 \right\}e_1(H) \\ &= \frac{n^2(13n^2 - 56n + 52)}{4(n-2)^2}H^3 + \frac{2n(n-1)}{n-2}H^2\beta + \frac{4(2n-3)}{n}H\beta^2, \\ & \left\{ \omega_{21}^2 + \omega_{31}^3 + \frac{n-8}{2}\omega_{41}^4 \right\}e_1(H) \\ &= \frac{n^2(3n+4)}{4(n-2)}H^3 - \frac{n^2}{n-2}H^2\beta + 2H\beta^2 - aH. \end{aligned}$$

These equations imply

$$\begin{aligned} & \frac{2(2n^3 + 31n^2 - 112n + 96)}{n^2} \omega_{41}^4 e_1(H) \\ &= \frac{7n^4 - 56n^3 + 86n^2 + 152n - 192}{2(n-2)^2} H^3 - \frac{11n^2 - 52n + 48}{n-2} H^2 \beta \\ &+ \frac{2(5n^2 - 44n + 48)}{n^2} H \beta^2 - \frac{a(9n^2 - 50n + 48)}{n^2} H, \end{aligned} \quad (3.49)$$

$$\begin{aligned} & \frac{2(2n^3 + 31n^2 - 112n + 96)}{n^2} (\omega_{21}^2 + \omega_{31}^3) e_1(H) \\ &= \frac{n^2(5n^3 - 82n^2 + 256n - 200)}{4(n-2)^2} H^3 + \frac{n(3n^2 - 16n + 16)}{2(n-2)} H^2 \beta \\ &+ \frac{3n^2 - 40n + 48}{n} H \beta^2 + \frac{a(n+2)}{2} H. \end{aligned} \quad (3.50)$$

After combining (3.50) with (3.42) and (3.37), we get

$$\begin{aligned} & \frac{2n^3 + 31n^2 - 112n + 96}{n(n-3)} \omega_{21}^2 \omega_{31}^3 \\ &= \frac{n^2(n^3 - 144n^2 + 480n - 392)}{4(n-2)^2} H^2 + \frac{n(n^3 - 56n^2 + 176n - 144)}{2(n-2)(n-3)} H \beta \\ &+ \frac{5n^3 - 18n^2 + 56n - 48}{n(n-3)} \beta^2 + \frac{a(n+2)}{2} H. \end{aligned} \quad (3.51)$$

Now taking into account (3.42), (3.49) becomes

$$\begin{aligned} & - \frac{4(2n^3 + 31n^2 - 112n + 96)}{n^3} (e_1(H))^2 \\ &= \frac{7n^4 - 56n^3 + 86n^2 + 152n - 192}{2(n-2)^2} H^4 - \frac{11n^2 - 52n + 48}{n-2} H^3 \beta \\ &+ \frac{2(5n^2 - 44n + 48)}{n^2} H^2 \beta^2 - \frac{a(9n^2 - 50n + 48)}{n^2} H^2. \end{aligned} \quad (3.52)$$

After differentiating (3.52) with respect to  $e_1$  and by using (3.49), (3.47), (3.41) and (3.6), we obtain

$$L(H, \beta) \omega_{21}^2 + M(H, \beta) \omega_{31}^3 = 0, \quad (3.53)$$

where  $L$  and  $M$  take the form

$$\begin{aligned} L(H, \beta) &= a_0 H^4 + a_1 H^3 \beta + a_2 H^2 \beta^2 + a_3 H \beta^3 + a_4 H^2 + a_5 H \beta, \\ M(H, \beta) &= b_0 H^4 + b_1 H^3 \beta + b_2 H^2 \beta^2 + b_3 H \beta^3 + b_4 H^2 + b_5 H \beta \end{aligned}$$

for some constants  $a_i$  and  $b_i$  with respect to  $n$ .

Moreover, substituting (3.41) into (3.50) yields

$$(\omega_{21}^2 + \omega_{31}^3) \left\{ (c_1 H - \beta) \omega_{21}^2 + [(c_1 - c_2) H + \beta] \omega_{31}^3 \right\} = N(H, \beta), \quad (3.54)$$

where

$$N(H, \beta) = c_0 H^3 + c_1 H^2 \beta + c_2 H \beta^2 + c_3 H$$

for some constants  $c_i$  with respect to  $n$ .

To eliminate  $\omega_{21}^2$  and  $\omega_{31}^3$  from (3.51), (3.53) and (3.54), a direct computation gives the following equation of ninth degree involving  $H$  and  $\beta$

$$\begin{aligned} & c_{90}H^9 + c_{81}H^8\beta + c_{72}H^7\beta^2 + c_{63}H^6\beta^3 + c_{54}H^5\beta^4 + c_{45}H^4\beta^5 \\ & + c_{36}H^3\beta^6 + c_{27}H^2\beta^7 + c_{18}H\beta^8 + c_{09}\beta^9 + c_{70}H^7 + c_{61}H^6\beta \\ & + c_{52}H^5\beta^2 + c_{43}H^4\beta^3 + c_{34}H^3\beta^4 + c_{25}H^2\beta^5 + c_{16}H\beta^6 + c_{07}\beta^7 \\ & + c_{50}H^5 + c_{41}H^4\beta + c_{32}H^3\beta^2 + c_{23}H^2\beta^3 + c_{14}H\beta^4 + c_{05}\beta^5 \\ & + c_{30}H^3 + c_{21}H^2\beta + c_{12}H\beta^2 + c_{03}\beta^3 = 0, \end{aligned} \quad (3.55)$$

where the coefficients  $c_{ij}$  ( $i, j = 0, \dots, 9$ ) are constants concerning  $n$ .

Note that  $\beta$  is not constant in general. In fact, if  $\beta$  is a constant, then (3.55) is an algebraic equation of  $H$  with constant coefficients. Thus the real function  $H$  must be a constant and hence the conclusion follows immediately.

Now, let us consider an integral curve of the vector field  $e_1$  passing through  $p = \gamma(t_0)$  as  $\gamma(t), t \in I$ . Because  $e_1(H), e_1(\beta) \neq 0$  and  $e_i(H) = e_i(\beta) = 0$  for  $2 \leq i \leq n$  according to Lemma 3.2, we may assume that  $t = t(\beta)$  and  $H = H(\beta)$  in some neighborhood of  $\beta_0 = \beta(t_0)$ . It follows from (3.6), (3.41) and (3.53) that

$$\begin{aligned} \frac{dH}{d\beta} &= \frac{dH}{dt} \frac{dt}{d\beta} = \frac{e_1(H)}{e_1(\beta)} \\ &= \frac{(c_1H - \beta)\omega_{21}^2 + [(c_1 - c_2)H + \beta]\omega_{31}^3}{c_2(c_1H - \beta)\omega_{21}^2} \\ &= \frac{1}{c_2} - \frac{[(c_1 - c_2)H + \beta]L}{c_2(c_1H - \beta)M}. \end{aligned} \quad (3.56)$$

By differentiating (3.55) with respect to  $\beta$ , together with (3.56), we get another algebraic equation of twelfth degree involving  $H$  and  $\beta$ :

$$\begin{aligned} & b_{12,0}H^{12} + b_{11,1}H^{11}\beta + b_{10,2}H^{10}\beta^2 + b_{93}H^9\beta^3 + b_{84}H^8\beta^4 + b_{75}H^7\beta^5 \\ & + b_{66}H^6\beta^6 + b_{57}H^5\beta^7 + b_{48}H^4\beta^8 + b_{39}H^3\beta^9 + b_{2,10}H^2\beta^{10} + b_{1,11}H\beta^{11} \\ & + b_{0,12}\beta^{12} + b_{10,0}H^{10} + b_{91}H^9\beta + b_{82}H^8\beta^2 + b_{73}H^7\beta^3 + b_{64}H^6\beta^4 \\ & + b_{55}H^5\beta^5 + b_{46}H^4\beta^6 + b_{37}H^3\beta^7 + b_{28}H^2\beta^8 + b_{19}H\beta^9 + b_{0,10}\beta^{10} + b_{80}H^8 \\ & + b_{71}H^7\beta + b_{62}H^6\beta^2 + b_{53}H^5\beta^3 + b_{44}H^4\beta^4 + b_{35}H^3\beta^5 + b_{26}H^2\beta^6 + b_{17}H\beta^7 \\ & + b_{08}\beta^8 + b_{60}H^6 + b_{51}H^5\beta + b_{42}H^4\beta^2 + b_{33}H^3\beta^3 + b_{24}H^2\beta^4 + b_{15}H\beta^5 \\ & + b_{06}\beta^6 + b_{40}H^4 + b_{31}H^3\beta + b_{22}H^2\beta^2 + b_{13}H\beta^3 + b_{04}\beta^4 = 0, \end{aligned} \quad (3.57)$$

where the coefficients  $b_{ij}$  ( $i, j = 0, \dots, 12$ ) are some constants concerning  $n$ .

We may rewrite (3.55) and (3.57) respectively in the following forms

$$\sum_{i=0}^9 q_i(H)\beta^i = 0, \quad \sum_{j=0}^{12} \bar{q}_j(H)\beta^j = 0, \quad (3.58)$$

where  $q_i(H)$  and  $\bar{q}_j(H)$  are polynomials functions of the mean curvature function  $H$ . After eliminating  $\beta$  between the two equations in (3.58), we obtain a non-trivial algebraic polynomial equation of  $H$  with constant coefficients. Hence we conclude that the real function  $H$  has to be a constant, which contradicts to our original

assumption. Therefore the mean curvature function  $H$  must be constant for any  $\delta(3)$ -ideal null 2-type hypersurfaces in a Euclidean space.

Consequently, we have proved the following theorem.

**Theorem 1.** *Every  $\delta(3)$ -ideal null 2-type hypersurface in a Euclidean space must have constant mean curvature and constant scalar curvature.*

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